

$$\text{Ex. } A_2 \begin{pmatrix} 1 & 2 & 0 & 1 & -1 \\ 2 & 1 & 3 & 1 & 0 \\ -1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 2 & 8 \end{pmatrix} \xrightarrow[\text{form}]{\text{echelon}} \begin{pmatrix} \textcircled{1} & 0 & 2 & 0 & -1 \\ 0 & \textcircled{1} & -1 & 0 & -2 \\ 0 & 0 & 0 & \textcircled{1} & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{Let } x_3 = s \quad x_5 = t$$

$$x_4 = -4t, \quad x_2 = s + 2t, \quad x_1 = t - 2s$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} t - 2s \\ s + 2t \\ s \\ -4t \\ t \end{pmatrix} = s \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ 0 \\ -4 \\ 1 \end{pmatrix}$$

$$\text{The Spanning set of the kernel} = \left\{ \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \\ -4 \\ 1 \end{pmatrix} \right\}$$

How to find the range of a matrix transformation

$$\begin{pmatrix} \textcircled{1} & 0 & 2 & 0 & -1 \\ 0 & \textcircled{1} & -1 & 0 & -2 \\ 0 & 0 & 0 & \textcircled{1} & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \text{from previous example}$$

$$\text{Spanning set of the range is } = \left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \\ 4 \\ 0 \end{pmatrix} \right\}$$

Same columns of the pivot
but from original matrix

Theorem: The pivot columns of a matrix A form a basis for the range of the linear transformation

$$\text{Ex. } A = \begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix} \xrightarrow[\text{form}]{\text{echelon}} \begin{pmatrix} \textcircled{1} & -2 & 0 & -1 & 3 \\ 0 & 0 & \textcircled{1} & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Basis of the range is $\left\{ \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix} \right\}$

The nullity and the rank of linear transformation

Let $T: V \rightarrow W$ be a linear transformation. The dimension of the kernel of T is called the nullity of T . The dimension of the range of T is called the rank of T .

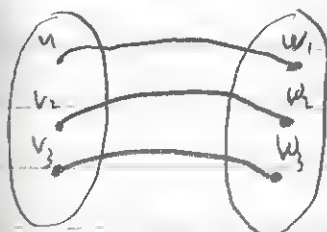
Special case: In case of matrix transformation $T(v) = Av$

$$\text{rank}(T) = \text{rank}(A)$$

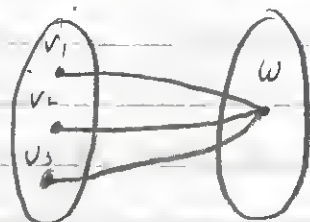
$$\text{nullity}(T) = \text{no. of free variables}$$

$$\text{Ex. } \begin{pmatrix} \textcircled{1} & 0 & 2 & 0 & -1 \\ 0 & \textcircled{1} & -1 & 0 & -2 \\ 0 & 0 & 0 & \textcircled{1} & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \text{from previous example} \\ \text{rank} = 3, \text{ nullity} = 2$$

One-to-one and onto linear transformation

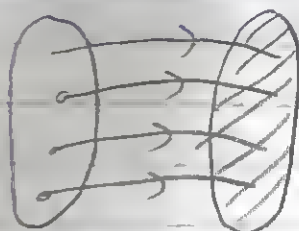


Domain Codomain
one-to-one



Domain Codomain
not one-to-one

if $V_1 \neq V_2 \iff \omega_1 \neq \omega_2$



Range = Codomain
(Onto)

Ex. $\begin{pmatrix} \textcircled{1} & -4 & 8 & 1 \\ 0 & \textcircled{2} & -1 & 3 \\ 0 & 0 & 0 & \textcircled{5} \end{pmatrix} \rightarrow$ echelon form

Consistent \Rightarrow onto (~~not one-to-one~~)
free variable exist \Rightarrow not one-to-one

Theorem: Let $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation. Then T is one-to-one if the equation $T(x) = \underline{0}$ has only the trivial solution [no free variables]

Theorem: Let $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation. Then T is onto if and only if the rank of T is equal to m (no. of rows)

Ex. a) $\begin{pmatrix} \textcircled{1} & 2 & 0 \\ 0 & \textcircled{1} & 1 \\ 0 & 0 & \textcircled{1} \end{pmatrix}$ one-to-one onto

b) $\begin{pmatrix} \textcircled{1} & 2 \\ 0 & \textcircled{1} \\ 0 & 0 \end{pmatrix}$ one-to-one not onto

c) $\begin{pmatrix} \textcircled{1} & 2 & 0 \\ 0 & \textcircled{1} & -1 \end{pmatrix}$ not one-to-one onto

d) $\begin{pmatrix} \textcircled{1} & 2 & 0 \\ 0 & \textcircled{1} & 1 \\ 0 & 0 & 0 \end{pmatrix}$ not one-to-one not onto